Array Interpolation Based on Multivariate Adaptive Regression Splines

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***Abstract*—Array processing is an important topic in the signal processing field. Many important signal processing techniques such as Spatial Smoothing, Forward Backward Averaging and Root-MUSIC, rely on antenna arrays with specific and precise structures. Arrays with such ideal structures, such as a centro-hermitian structure, are often hard to build in practice. Array interpolation is used to enable the usage of these techniques with imperfect (not having a centro-hermitian structure) arrays. Most interpolation methods rely on methods based on least squares (LS) to map the output of a perfect virtual array based on the real array. In this work, the usage of Multivariate Adaptive Regression Splines (MARS) is proposed instead of the traditional LS to interpolate arrays with responses largely different from the ideal using non-linear mapping functions.**

***Index Terms*—array interpolation, multivariate adaptive regression splines**

I. Introduction

Antenna arrays are employed in a variety of signal processing algorithms to estimate the direction of arrival (DOA) of received signals. Ideally, these antennas should provide isotropic responses in amplitude and in phase for their field of view. For directional antennas, this is a somewhat achievable goal. However, in order to estimate DOAs of incoming signals, the antennas should ideally be isotropic.

Besides requiring isotropic responses, many DOA estimation techniques such as IQML [1], Root-WSF [2] and Root-MUSIC [3] rely on a Vandermonde or centro-hermitian array response. Spatial smoothing (SPS) [4] and forward backward averaging (FBA) [5] also require an array with a Vandermonde and centro-hermitian response, respectively. On the other hand, Estimation of Signal Parameters via Rotational Invariance (ESPRIT) [6] demands a shift invariant array response.

Arrays with perfect Vandermonde, centro-hermitian, shift invariant, or even isotropic responses are hard to obtain in practice. Each antenna element has a unique response and the antenna's response can also be affected by mutual coupling. In order to mitigate the effects of imperfect array responses, the process of transforming real (i.e., not having the desired response) array responses into desired virtual ideal array responses by means of a model is also known as array interpolation [7].

Different array interpolation methods have been proposed to allow DOA estimation algorithms such as MUSIC [8] and Root-MUSIC [3], [9] to be used in non-ideal arrays. Alternatives that allow the application of ESPRIT algorithm have been presented in [10] and [11], however, these alternatives do not allow SPS and FBA to be applied. In [12], an alternative for applying ESPRIT with FBA and SPS was first presented. This work was extended to the multidimensional case in [13].

All of the mentioned interpolation methods apply a least squares (LS) step for building the model that represents the virtual array based on the inputs of the real array. The LS method is capable of providing good results for arrays with responses close to the desired one such that the mapping between the real and virtual arrays can be approximated linearly. However, a linear approximation might not be enough to map the real response onto the virtual response with tolerable transformation errors.

To tackle arrays with challenging geometries or challenging responses a method capable of expressing non-linear relationships between the real and the desired response is necessary. This work proposes the application of an adaptive modeling method which is capable of finding and properly modeling non-linear mappings between real and virtual arrays in the form of the Multivariate Adaptive Regression Splines (MARS) [14] method. While the MARS approach offers benefits for arrays with responses that strongly differ from the desired response, it is computationally expensive and does not offer large performance improvements over the linear approach for arrays with a response close to the desired one. Therefore, this work also proposes a method of selecting between the linear and MARS approach based on the transformation error of the linear approach.

The remainder of this work is divided as follows. Section II presents the data model. Section III briefly details the

LS method for array interpolation. Section IV presents the proposed MARS interpolation approach. Section V presents a set of numerical simulations and discusses their results. Finally, conclusions are drawn in Section VI.

II. Data Model

We consider a set of d wavefronts impinging onto an antenna array composed of M antenna elements. The received baseband signal can be expressed in matrix form as

$$\mathbf{X} = \mathbf{AS} + \mathbf{N} \in \mathbb{C}^{M \times N}, \tag{1}$$

where $\mathbf{S} \in \mathbb{C}^{d \times N}$ is the matrix containing the N symbols transmitted by each of the d sources, $\mathbf{N} \in \mathbb{C}^{M \times N}$ is the noise matrix with its entries drawn from $\mathcal{CN}(0, \sigma_n^2)$, and

$$\mathbf{A} = [\mathbf{a}(\theta_1), \mathbf{a}(\theta_2), ..., \mathbf{a}(\theta_d)] \in \mathbb{C}^{M \times d}, \tag{2}$$

where θ_i is the azimuth angle of the i-th signal and $\mathbf{a}(\theta_i) \in \mathbb{C}^{M \times 1}$ is the array response (empirical measurement).

The received signal covariance matrix $\mathbf{R_{XX}} \in \mathbb{C}^{M \times M}$ is given by

$$\mathbf{R_{XX}} = \mathrm{E}\{\mathbf{XX}^{\mathrm{H}}\} = \mathbf{AR_{SS}A}^{\mathrm{H}} + \mathbf{R_{NN}}, \tag{3}$$

where $(\cdot)^{\mathrm{H}}$ stands for the conjugate transposition, $E\{\cdot\}$ is the expectation operator, and

$$\mathbf{R_{SS}} = \begin{bmatrix} \sigma_1^2 & \gamma_{1,2}\sigma_1\sigma_2 & \cdots & \gamma_{1,d}\sigma_1\sigma_d \\ \gamma_{1,2}^*\sigma_1\sigma_2 & \sigma_2^2 & & \vdots \\ \vdots & & \ddots & \\ \gamma_{1,d}^*\sigma_1\sigma_d & \gamma_{2,d}^*\sigma_2\sigma_d & \cdots & \sigma_d^2 \end{bmatrix}, \tag{4}$$

where σ_i^2 is the power of the $i-$th signal and $\gamma_{a,b} \in \mathbb{C}$, $|\gamma_{a,b}| \leq 1$ is the cross correlation coefficient between signals a and b. $\mathbf{R_{NN}} \in \mathbb{C}^{M \times M}$ is a matrix with σ_n^2 over its diagonal and zeros elsewhere. An estimate of the signal covariance matrix can be obtained by

$$\hat{\mathbf{R}}_{\mathbf{XX}} = \frac{\mathbf{XX}^{\mathrm{H}}}{N}. \tag{5}$$

III. Linear Array Interpolation

Array interpolation aims to estimate what signal would have been received at an anenna array with a specific desired geometry based on the signal that was received by a real antenna array . Linear array interpolation is usually done using a LS approach. The problem is set up as finding a transformation matrix $\mathbf{B}$ that is given by

$$\mathbf{BA}_{\mathcal{S}} = \bar{\mathbf{A}}_{\mathcal{S}}, \tag{6}$$

where $\mathbf{A}_{\mathcal{S}}$ and $\bar{\mathbf{A}}_{\mathcal{S}}$ are real and virtual array responses for a given countable and discrete set of angles $\mathcal{S}$, respectively. Applying matrix $\mathbf{B}$ to a snapshot of received data $\mathbf{y} \in \mathbb{C}^{M \times 1}$ can be done with a simple matrix multiplication, which is equivalent to applying a linear model for each of the outputs of the virtual antenna array. This linear model can be given

$$[\mathbf{y}]_m = [\mathbf{B}]_{1,m}[\mathbf{y}]_m + [\mathbf{B}]_{2,m}[\mathbf{y}]_m + \ldots + [\mathbf{B}]_{M,m}[\mathbf{y}]_m, \tag{7}$$

where $[\cdot]_{i,j}$ is the element of a matrix indexed by i and j, and $m \in \{1, \ldots M\}$.

This model is usually not capable of transforming the response perfectly across the entire filed of view except for the case where a large number of antenna elements is present or a very small sector is used. Large transformation errors often result in a large bias on the final DOA estimation, thus, usually, the response region is divided into a set of regions called sectors, and a different transform matrix is set up for each sector (sector-by-sector processing).

IV. MARS Array Interpolation

For small sectors and arrays that have real responses that do not differ strongly from the desired response, linear array interpolation is capable of providing good results [15]. However, as the size of the sectors increases or as the real responses differs strongly from the desired ones, the linear approach might not be sufficient. For such cases, this work proposes the application of a MARS approach to build a model between the real and desired responses.

MARS was proposed in [14] as a non-parametric regression method that extended previous step-wise linear regression methods using splines. As a non-parametric regression method, MARS does not require any knowledge of the relationship between the predictor and predicted data. It derives all of its information from the data set itself and is, therefore, a flexible regression method. The MARS model relies on functions known as hinge functions. Hinge functions take the form

$$h(k, z) = \begin{cases} \max(0, k - z) \\ \text{or} \\ \max(0, z - k) \end{cases} \tag{8}$$

where

$$\max(a, b) = \begin{array}{l} a \text{ if } a > b \\ b \text{ if } a \leq b \end{array}, \tag{9}$$

and k is known as a knot. Figure 1 exemplifies the two possible forms of a hinge function with $k = 3$.

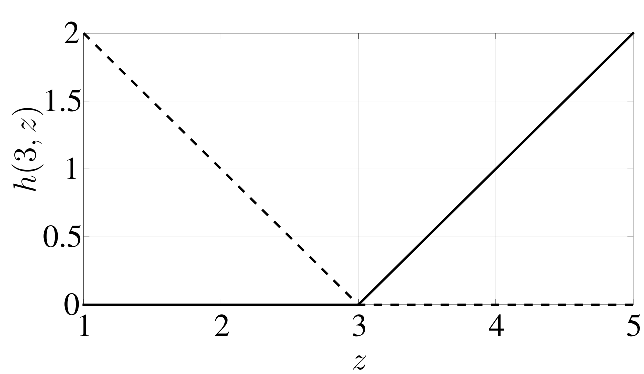

Fig. 1: Example of hinge functions $\max(0, 3 - z)$ and $\max(0, z - 3)$

While a hinge function is piece-wise linear, the MARS model allows hinge functions to be multiplied together. Therefore, the MARS model is capable of taking into account non-linear relationships between the input and output variables. Figure 2 shows how a non-linear relationship is created from the multiplication of hinge functions $h(1,z)$, $h(2,z)$, and $h(3,z)$.

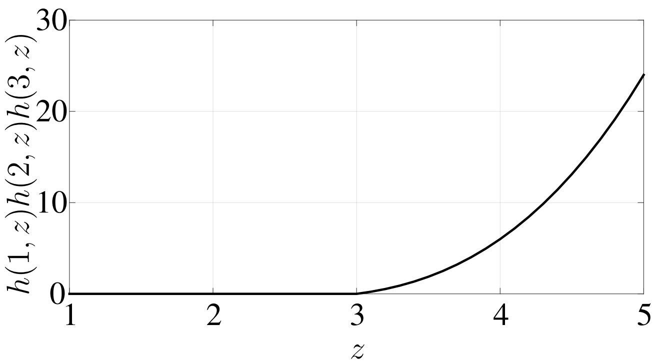

Fig. 2: Example of a non-linear relationship modeled by a multiplication of hinge functions

The proposed MARS interpolation approach builds a pair of models b_m^{R} and b_m^{I} for the real and imaginary parts of the response of each of the antennas of the array. These models can be written as a linear combination of basis functions

$$b_m^{\mathrm{R}}(\mathbf{a}_s(\theta)) = \sum_{l=1}^{L_m^R} c_{l_m}^R F_{l_m}^R(\mathbf{a}_s(\theta)), \tag{10}$$
$$b_m^{\mathrm{I}}(\mathbf{a}_s(\theta)) = \sum_{l=1}^{L_m^I} c_{l_m}^I F_{l_m}^I(\mathbf{a}_s(\theta)),$$

where $\theta_s \in \mathcal{S}$, $s = 1, \ldots, |\mathcal{S}|$, $m = 1, \ldots, M$, $c_{l_m}^R$ and $c_{l_m}^I$ are weighting constants, $F_{l_m}^R$ and $F_{l_m}^I$ are basis functions that represent either a hinger function or a multiplication of hinge functions yielding a non-linear relationship. That is, if the relationship between the real and virtual arrays can be expressed linearly, $F_{l_m}^R$ and $F_{l_m}^I$ are always a single hinge function. If the relationship is non-linear, at least of of the $F_{l_m}^R$ and $F_{l_m}^I$ are is a multiplication of hinge functions. Each model has a different number of L_m^R and L_m^I functions and weighting constants.

Building a MARS model relies only on the data set and is done recursively using a two step algorithm. In the first step, the algorithm will introduce basis functions to the model. Basis functions are added depending on their effect on the so-called goodness-of-fit, where the basis function which impacts the goodness-of-fit the most is introduced first. This step is repeated until a bound on the model complexity is reached. This bound is usually set so that the model generated on the first step is over fitted.

In the second step, the complexity of the model is reduced by removing the basis functions that have the least impact on the accuracy of the model. The second step is important since it reduces over fitting and leads to a more general model. The models are compared using the Generalized Cross-Validation (GCV), which is based in the mean-squared residual error and a penalty based on model complexity. The GCV is given by

$$\mathrm{GCVR} = \frac{\frac{1}{|\mathcal{S}|}\sum_{s=1}^{|\mathcal{S}|}(\Re\{[\bar{\mathrm{a}}(\theta_s)]_m\} - b_m^R(\mathrm{a}(\theta_s)))^2}{(1 - \frac{L_m^R + p\frac{L_m^R - 1}{2}}{|\mathcal{S}|})^2}, \tag{11}$$

$$\mathrm{GCVI} = \frac{\frac{1}{|\mathcal{S}|}\sum_{s=1}^{|\mathcal{S}|}(\Im\{[\bar{\mathrm{a}}(\theta_s)]_m\} - b_m^I(\mathrm{a}(\theta_s)))^2}{(1 - \frac{L_m^I + p\frac{L_m^I - 1}{2}}{|\mathcal{S}|})^2}, \tag{12}$$

where, b_m is a model, for either the real or imaginary parts of the response of the m-th antenna and $[\bar{\mathbf{a}}_s(\theta)]_m$ is the corresponding real or imaginary part of the m-th element of the steering vector $\bar{\mathbf{a}}_s(\theta)$. $|\mathcal{S}|$ is the number of steering vectors used to build the model, C is the number of basis functions used for the model and p is a penalty factor. Different values for p are discussed in [14], where a value of 3 is suggested as a default value.

Since a model is built for the real and imaginary parts of the response of each of the antennas, a total of $2M$ models need to be created to interpolate the array using the MARS approach. This process is computationally complex and can be very demanding for large arrays. To use this approach in an online system the models can be built in a initialization step. Sectors of interest for the application of the MARS interpolation can be selected and a model can be built only for such sectors. Furthermore, in most cases, the antenna array will have a response which is similar enough to the desired response in some of its field of view so that LS approach yields good results. Thus, the MARS approach can be used only on portions of the field of view where the real response is varies strongly from the desired one.

A possible way of deciding between the linear and the MARS approach is by looking into the transformation error when employing the linear approach. The error $\epsilon(\mathcal{S})$ of the transform for a given sector $\mathcal{S}$ is defined as

$$\epsilon(\mathcal{S}) = \frac{\left\|\bar{\mathbf{A}}_{\mathcal{S}} - \mathbf{B}\mathbf{A}_{\mathcal{S}}\right\|_{\mathrm{F}}}{\left\|\bar{\mathbf{A}}_{\mathcal{S}}\right\|_{\mathrm{F}}} \in \mathbb{R}^+. \tag{13}$$

Large transformation errors often result in a large bias on the final DOA estimations. In order to reduce the transformation error the size of the sectors used to calculate the transformation matrix can be reduced. This, however, may not always be possible and may not always be enough to reduce the transformation error to an acceptable threshold. In such cases, the MARS approach can be employed.

The MARS approach is specially useful for arrays with a small number of elements. As shown in (6), the degrees of freedom of the linear approach are limited by the number of antennas in the virtual array. It is possible to increase the degrees of freedom by creating a virtual array with a larger number of antennas, however, this will lead to a transformation matrix that is ill conditioned and result in a large bias in the DOA estimates. The MARS approach, on the other hand, can build as many hinge functions and relationships between them as necessary. While the MARS will still benefit from having a larger array, since it provides more input variables for the model, it is less sensitive to a reduction in the size of the array.

In order to properly estimate the DOAs the noise covariance after the transformation needs to be known. This requires that either the noise covariance prior to the estimation is known, or that noise samples can be obtained prior to the transmission of the signals of interest in order to estimate the noise covariance at the antenna array. Once the noise covariance is known, the MARS model can be applied in order to estimate the noise covariance $\bar{\mathbf{R}}_{\mathbf{NN}}$ at the output of the transformation. The signal subspace can then be estimated by applying the generalized eigen value decomposition (GEVD) on the covariance matrix $\bar{\mathbf{R}}_{\mathbf{XX}}$, which is the covariance matrix of the interpolated signal:

$$\bar{\mathbf{R}}_{\mathbf{XX}}\mathbf{E} = \bar{\mathbf{R}}_{\mathbf{NN}}\mathbf{E}\boldsymbol{\Lambda}, \tag{14}$$

where $\mathbf{E} \in \mathbb{C}^{M\times M}$ is a matrix containing the generalized eigenvectors and $\boldsymbol{\Lambda} \in \mathbb{R}^{M\times M}$ is a matrix containing the generalized eigenvalues in its diagonal. By selecting the eigenvectors related to the $\hat{d}$ largest eigenvalues the so called signal subspace $\mathbf{E}_s \in \mathbb{C}^{M\times \hat{d}}$ is constructed. This signal subspace needs to be dewhitened or projected back onto the original response subspace prior to estimation, this can be done by

$$\bar{\mathbf{E}}_s = \bar{\mathbf{R}}_{\mathbf{NN}}\mathbf{E}_s. \tag{15}$$

With this subspace estimate at hand, Total Least Squares (TLS) ESPRIT [6] may be applied.

V. Numerical Results

The array response assumed in the simulations in this work is constructed by randomly displacing the elements of a uniform linear array (ULA). The array has inner element spacing of $b = \frac{\lambda}{2}$ and the elements are displaced to a point belonging to a circle with center on the original antenna position and radius $a = \frac{\epsilon\lambda}{2}$, where ϵ is the displacement error in fractions of the wavelength λ of the carrier frequency of the signal, as shown in Figure 3. For obtaining $\hat{\mathbf{R}}_{\mathbf{XX}}$ we use $N = 100$ snapshots and the root mean squared error (RMSE) for DOA estimation is calculated with respect to $K = 1000$ Monte Carlo simulations. Two signals impinging from $\theta_1 = 45°$ and $\theta_2 = 15°$ with $\sigma_1^2 = \sigma_2^2 = 1$ and $\gamma_{1,2} = 1$ according to (4) are impinging on the array. FBA and SPS are used to decorrelate the received signals prior to the DOA estimation. The Signal to Noise Ratio (SNR) is defined as $\mathrm{SNR} = \frac{\sigma_1^2+\sigma_2^2}{\sigma_{\mathrm{n}}^2}$. The RMSE can be given as

$$\mathrm{RMSE} = \sqrt{\frac{1}{K}\sum_{k=1}^{K}\left((\hat{\theta}_{1,k}-\theta_1)^2 + (\hat{\theta}_{2,k}-\theta_2)^2\right)}, \tag{16}$$

where $\hat{\theta}_{i,k}$ is the estimate of θ_i at the k-th Monte Carlo run.

The first metric analyzed is the behavior of the MARS and linear approaches as the displacement error increases. Figure 4 shows the results for ϵ varying from 0.02 to 0.25 and a SNR of 35 dB.

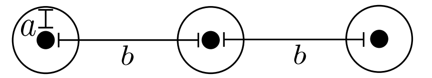

Fig. 3: Simulation array setup

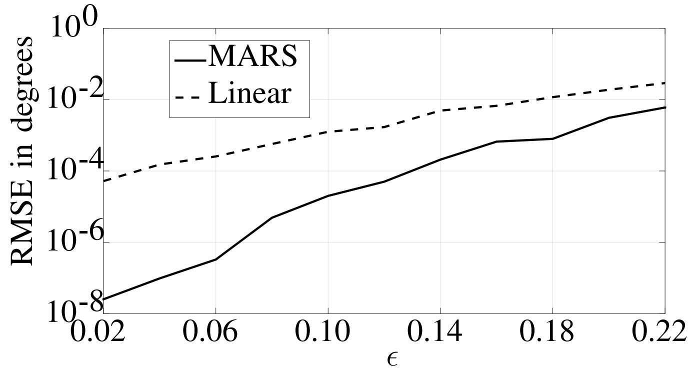

Fig. 4: Average of LS and MARS for varying ϵ

The results show that for small displacement errors the MARS approach and the linear approach provide similar results (please consider that the y-axis is in logarithmic scale). Furthermore, as the displacement increases the bias for the LS method increases much more rapidly than that of the MARS approach since, for large displacement errors, a linear relationship is not sufficient to create a proper model between the real and virtual arrays.

Figure 5 presents the results for a varying number of antennas with $\epsilon = 0.12$. The results show that the MARS method outperforms the LS method, especially in scenarios where the number of antennas of the array is small. This is also a beneficial scenario for MARS since it requires less models to be created, therefore reducing the computational load.

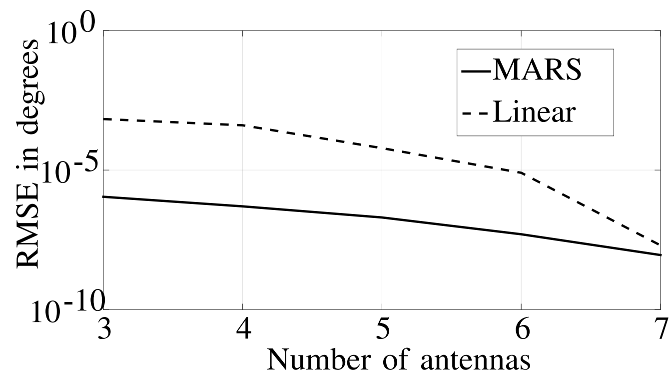

Fig. 5: Average performance of LS and MARS for varying number of antennas and $\epsilon = 0.12$

VI. Conclusion

This work proposed a way of performing array interpolation using the MARS method. MARS is capable of expressing non-linear relationships between the real and the desired response. This work also discusses a method of deciding between MARS and linear interpolation. While MARS is capable of providing an improvement of the linear interpolation, it is computationally expensive and does not always provide a large improvement.

Numerical simulations were performed aiming to study the performance of the proposed interpolation method. The first set of simulations highlighted the benefits of MARS in terms of DOA estimation bias when the real array response strongly differs from the desired virtual response. For large displacement errors in the array, the MARS approach achieved a significant improvement in DOA estimation accuracy. A second set of simulations highlighted the benefits of MARS over the linear approach in scenarios were the array size is small.

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REFERENCES

[1] Y. Bresler and A. Macovski, "Exact Maximum Likelihood Estimation of Superimposed Exponentials Signals in Noise," *IEEE ASSP Magazine*, vol. 34, pp. 1081–189, 1986.

[2] P. Stoica and A. Nehorai, "A Novel Eigenanalysis Method for Direction Estimation," in *Proceedings IEEE F.*, 1990.

[3] A. J. Barabell, "Improving the Resolution Performance of Eigenstructured Based Direction-Finding Algorithms," in *Proceedings of ICASSP 83*, 1983.

[4] J. E. Evans, J. R. Johnson, and D. F. Sun, "Application of advanced signal processing techniques to angle of arrival estimation in ATC navigation and surveillance system," Massachusetts Institute of Technology, Tech. Rep., 1982.

[5] S. Pillai and B. H. Kwon, "Forward/backward Spatial Smoothing Techniques for Coherent Signal Identification ," *IEEE Transactions on Acoustics, Speech and Signal Processing*, vol. 37, pp. 8–9, January 1989.

[6] R. Roy and T. Kailath, "ESPRIT - estimation of signal parameters via rotation invariance techniques," *IEEE Transactions on Acoustics Speech and Signal Processing*, vol. 17, 1989.

[7] T. Bronez, "Sector interpolation of non-uniform arrays for efficient high resolution bearing estimation," in *Acoustics, Speech, and Signal Processing, 1988. ICASSP-88., 1988 International Conference on*, 1988.

[8] R. O. Schmidt, "Multiple emitter location and signal parameter estimation," *IEEE Transactions on Antennas and Propagation*, vol. 34, pp. 276–280, 1986.

[9] B. Friedlander, "The root-MUSIC algorithm for direction finding with interpolated arrays," *Signal Processing*, vol. 30, pp. 15–29, 1993.

[10] M. Buhren, M. Pesavento, and J. F. Bohme, "A new approach to array interpolation by generation of artificial shift invariances: interpolated ESPRIT," in *Acoustics, Speech, and Signal Processing, 2003. Proceedings. (ICASSP '03). 2003 IEEE International Conference on*, 2003.

[11] A. Weiss and M. Gavish, "The interpolated ESPRIT algorithm for direction finding," in *Electrical and Electronics Engineers in Israel, 1991. Proceedings., 17th Convention of*, 1991.

[12] M. A. M. Marinho, F. Antreich, J. P. C. L. da Costa, and J. A. Nossek, "A Signal Adaptive Array Interpolation Approach with Reduced Transformation Bias for DOA estimation of Highly Correlated Signals," in *Acoustics, Speech, and Signal Processing, 2004. Proceedings. (ICASSP '14). IEEE International Conference on*, 2014.

[13] M. A. M. Marinho, J. P. C. L. da Costa, F. Antreich, and L. R. de Menezes, "Multidimensional array interpolation applied to direction of arrival estimation," in *19th International ITG Workshop on Smart Antennas (WSA 2015)*, 2015.

[14] J. H. Friedman, "Multivariate adaptive regression splines," *The annals of statistics*, pp. 1–67, 1991.

[15] B. Lau, M. Viberg, and Y. Leung, "Data-adaptive array interpolation for DOA estimation in correlated signal environments," in *Acoustics, Speech, and Signal Processing, 2005. Proceedings. (ICASSP '05). IEEE International Conference on*, 2005.