UNSCENTED TRANSFORMATION BASED ARRAY INTERPOLATION

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ABSTRACT

It is impossible to enforce exact responses for each sensor involved in an antenna array. Important signal processing techniques such as Estimation of Signal Parameters via Rotational Invariance (ESPRIT), Forward Backward Average (FBA) and Spatial Smoothing (SPS) rely on sensor arrays with Vandermonde or centro-hermitian responses. To achieve such responses array interpolation is often necessary. In this work a novel way of performing array interpolation while minimizing the transformation error using the Unscented Transformation (UT) is presented. The UT provides a different method for mapping interpolated regions and also exhibits a new insight into array interpolation and its current limitations. A set of numerical simulations presents promising results for array interpolation employing the UT.

***Index Terms*—** Array Interpolation, Array Mapping, Antenna Arrays, Unscented Transformation

1. INTRODUCTION

Direction of arrival (DOA) estimation methods such as Iterative Quadratic Maximum Likelihood (IQML) [1], Root-WSF [2] and Root-MUSIC [3] rely on arrays with specific responses, Vandermonde or centro-hermitian responses. Such methods offer benefits over methods which do not require a specific array geometry such as Maximum Likelihood (ML) [4, 5] or its iterative implementations such as the Expectation Maximization (EM) [6, 7, 8] and the Space Alternating Generalized Expectation Maximization (SAGE) [9, 10, 11] methods. Another important use of arrays with Vandermonde or centro-hermitian responses is that they provide the mathematical model necessary for the application of important tools such as Spatial Smoothing (SPS) [12] and Forward Backward Averaging (FBA) [13].

SPS and FBA are capable of providing "decorrelation" of received signals. Correlated signals are very common in strong multi-path scenarios, where copies of the same signal are received at nearly the same time instance from different reflection or diffraction points. Correlated signals heavily impact the performance of important array signal processing algorithms. For instance, model order selection methods [14, 15, 16, 17], used to estimate the number of received signals, have their performance highly impacted by the presence of correlated signals.

Exact Vandermonde and centro-hermitian responses are very hard to achieve in real sensor array implementations. In [18] a solution for mapping real and imperfect array responses into precise and desired responses was presented, such mapping is known as array interpolation. Array interpolation schemes rely on dividing the field of view of the array response into smaller angular regions, called sectors. For each of these sectors a transformation matrix is calculated using the empirical knowledge of the real array response. The larger the region transformed the larger will be the bias introduced due to transformation imprecision. After the transformation, FBA or SPS [19] and DOA estimation algorithms such as Root-MUSIC [20] can be applied.

One problem with the approach used in [19, 20, 21, 22], is that there is no guarantee with respect to what happens with signals received from outside the angular region to which the transformation matrix was calculated (out-of-sector signals). If the out-of-sector signal is correlated with any possible in-sector signals they introduces a large bias in the DOA estimation. In [23] and [24] this problem was first addressed by proposing a way to control the response of out-of-sector signals.

A problem introduced when any set of noisy data is transformed is that the noise changes its statistical properties. This problem can be averted by applying a pre-whitening step before DOA estimation algorithms such as MUSIC [25] and Root-MUSIC [3, 20]. Alternatives that allows the application of Estimation of Signal Parameters via Rotational Invariance (ESPRIT) algorithm [26] have been presented in [27] and [28], however, these alternatives do not allow SPS and FBA to be applied. In [29] an alternative for applying ESPRIT with FBA and SPS was first presented, relying on a signal based sector construction and discretization. This work was extended to the multidimensional case in [30].

All the mentioned array interpolation approaches present a common problem, which is the in-sector and, when treated, the out-of-sector regions that are discretized in order to calculate the transformation matrix. To the best of our knowledge, the discretization step is, for the most part, not clearly discussed and sometimes this step is only shown within the general context. In this work we propose the usage of the Unscented Transformation (UT) to systematically discretize several sectors of the field of view of the array in order to derive the transformation matrix.

The UT is a powerful tool used to transform a continuous probability density function (PDF) into a discrete version of itself, i.e. a probability probability mass function (PMF), while preserving the moments of the distribution. The UT is a fairly recent tool first presented in [31] for improving Kalman filters in nonlinear scenarios, and since then it has been applied in a variety of fields withing electrical engineering with very promising results [32, 33, 34, 35].

The remainder of this work is divided into five sections. In Section 2 the data model is presented. Section 3 presents the fundamentals of array interpolation. Section 4 describes the proposed interpolation method, whereas results from numerical simulations are shown in Section 5. Finally, in Section 6 conclusions are drawn.

2. DATA MODEL

We consider a set of d wavefronts impinging onto an antenna array composed of M antenna elements. The received baseband signal can be expressed in matrix form as

$$\mathbf{X} = \mathbf{AS} + \mathbf{N} \in \mathbb{C}^{M\times N}, \tag{1}$$

where $\mathbf{S} \in \mathbb{C}^{d\times N}$ is the matrix containing the N symbols transmitted by each of the d sources, $\mathbf{N} \in \mathbb{C}^{M\times N}$ is the noise matrix with its entries drawn from $\mathcal{CN}(0, \sigma_n^2)$, and

$$\mathbf{A} = [\mathbf{a}(\theta_1), \mathbf{a}(\theta_2), ..., \mathbf{a}(\theta_d)] \in \mathbb{C}^{M\times d}, \tag{2}$$

where θ_i is the direction of arrival of the $i-$th signal and $\mathbf{a}(\theta_i) \in \mathbb{C}^{M\times 1}$ is the array response.

The received signal covariance matrix $\mathbf{R_{XX}} \in \mathbb{C}^{M\times M}$ is given by

$$\mathbf{R_{XX}} = \mathrm{E}\{\mathbf{XX}^{\mathrm{H}}\} = \mathbf{AR_{SS}A}^{\mathrm{H}} + \mathbf{R_{NN}} \in \mathbb{C}^{M\times M}, \tag{3}$$

where $(\mathbf{.})^{\mathrm{H}}$ denotes the conjugate transposition, and

$$\mathbf{R_{SS}} = \begin{bmatrix} \sigma_1^2 & \gamma_{1,2}\sigma_1\sigma_2 & \cdots & \gamma_{1,d}\sigma_1\sigma_d \\ \gamma_{1,2}^*\sigma_1\sigma_2 & \sigma_2^2 & & \vdots \\ \vdots & & \ddots & \\ \gamma_{1,d}^*\sigma_1\sigma_d & \gamma_{2,d}^*\sigma_2\sigma_d & \cdots & \sigma_d^2 \end{bmatrix} \in \mathbb{C}^{d\times d}, \tag{4}$$

where σ_i^2 is the power of the $i-$th signal and $\gamma_{a,b} \in \mathbb{C}$, $|\gamma_{a,b}| \leq 1$ is the cross correlation coefficient between signals a and b. $\mathbf{R_{NN}} \in \mathbb{C}^{M\times M}$ is a diagonal matrix with σ_n^2 filling its diagonal. An estimate of the received signal covariance matrix can be obtained by

$$\hat{\mathbf{R}}_{\mathbf{XX}} = \frac{\mathbf{XX}^{\mathrm{H}}}{N} \in \mathbb{C}^{M\times M}. \tag{5}$$

3. CLASSICAL ARRAY INTERPOLATION

The array interpolation technique consists of finding a transformation matrix $\mathbf{B}$ that transforms the real array response $\mathbf{A}_{\mathcal{S}}$ for a given countable and discrete set of angles $\mathcal{S}$, called a sector, into the desired array response $\bar{\mathbf{A}}_{\mathcal{S}}$. Thus, the matrix $\mathbf{B}$ can be seen as the matrix that achieves the best transform between a set of vectors $\mathbf{A}_{\mathcal{S}}$ and $\bar{\mathbf{A}}_{\mathcal{S}}$. The simplest solution for obtaining $\mathbf{B}$ is a least squares fit via

$$\mathbf{B} = \bar{\mathbf{A}}_{\mathcal{S}}\mathbf{A}_{\mathcal{S}}^{\dagger} \in \mathbb{C}^{M\times M}, \tag{6}$$

where $(\mathbf{.})^{\dagger}$ stands for the Moore–Penrose pseudo-inverse. The transformation matrix $\mathbf{B}$, however, is usually not capable of transforming the response perfectly across the entire sector $\mathcal{S}$ except for the case where a large number of antenna elements is present or a very small sector is used. The error of the transform is defined as

$$\epsilon(\mathcal{S}) = \frac{\left\|\bar{\mathbf{A}}_{\mathcal{S}} - \mathbf{B}\mathbf{A}_{\mathcal{S}}\right\|_{\mathrm{F}}}{\left\|\bar{\mathbf{A}}_{\mathcal{S}}\right\|_{\mathrm{F}}} \in \mathbb{R}^{+}. \tag{7}$$

Large transformation errors often result in a large bias on the final DOA estimations, thus, usually, the response region is divided into a set of regions called sectors, and a different transform matrix is set up for each sector (sector-by-sector processing).

4. UNSCENTED TRANSFORMATION (UT) ARRAY INTERPOLATION

In this section the proposed method for array interpolation utilizing the UT is described in detail. Subsection 4.1 overviews the UT. Subsection 4.2 presents the application of the UT to the discretization of sectors for the calculation of the transformation matrix. In Subsection 4.3 the calculation of the transformation matrix using the UT is described. In Subsection 4.4 the data transformation is presented. Finally, in Subsection 4.5 the DOA estimation is shown.

4.1. Unscented Transformation (UT)

Although originally proposed for estimating the results of a nonlinear mapping of a probability distribution function, the UT has a very large number of applications and can be employed to reduce the problem of dealing with continuous PDF to dealing with a, usually much simpler, PMF.

The UT transforms a PDF into a PMF while retaining its statistical properties, i.e, its moments are preserved. The k-th pure moment of a random variable $\tilde{r}$ can be written as

$$\mathrm{E}\{r^k\} = \int_{\mathbb{R}} r^k p_{\tilde{r}}(r)dr = \sum_{j=1}^{J} w_j S_j, \in \mathbb{R} \tag{8}$$

where $p_{\tilde{r}}(r)$ is the probability that $\tilde{r}$ assumes the value r, w_j is the so called weight of the i-th sigma point S_j. The UT is a representation of a PDF into a PFM defined by the sigma points S_j and its weights w_j. Calculating the sigma points and its weights can be done if one has prior knowledge of the moments of the original PDF as

$$\sum_{j=1}^{k-1} w_j S_j^{k-1} = w_1 S_1^{k-1} + ... + w_{k-1} S_{k-1}^{k-1} = \mathrm{E}\{r^k\}. \tag{9}$$

Equation (9) shows that, in order to preserve the characteristics of $\tilde{r}$ up to the k-th moment, it is necessary to calculate $k-1$ sigma points and its weights by solving a nonlinear system of equations. Thus, there is a trade-off between simplicity in the calculation and the accuracy of the representation of higher order moments of the original PMF.

4.2. Sector Discretization

Classic interpolation relies on transforming an array response over a discrete set of angles $\mathcal{S}$. The choice of points that belong to $\mathcal{S}$ is usually an arbitrary one, based solely on the predefined sector bounds and the chosen angular resolution. We propose selecting the angles that belong to $\mathcal{S}$ by applying the UT.

Similar to [29], the first step is to look at the power received over the field of view of the array. For this step the conventional beamformer [36] can be applied, obtaining

$$\mathcal{L}(\theta|\mathbf{X}) = \frac{\mathbf{w}^{\mathrm{H}}(\theta)\hat{\mathbf{R}}_{\mathbf{XX}}\mathbf{w}(\theta)}{\mathbf{w}^{\mathrm{H}}(\theta)\mathbf{w}(\theta)\mathrm{tr}(\hat{\mathbf{R}}_{\mathbf{XX}})} \in \mathbb{R} \tag{10}$$

where $\hat{\mathbf{R}}_{\mathbf{XX}} = \frac{\mathbf{XX}^{\mathrm{H}}}{N}$ is the estimate of the signal covariance matrix, $\mathbf{w}(\theta) = \mathbf{a}(\theta_i)$ for $i = 1, \ldots, d$ and $\mathrm{tr}(\cdot)$ is the trace operation. Normalizing (10) we obtain the estimate of the likelihood $\mathcal{L}(\theta|\mathbf{X})$, which represents the probability that a signal is arriving from the DOA θ.

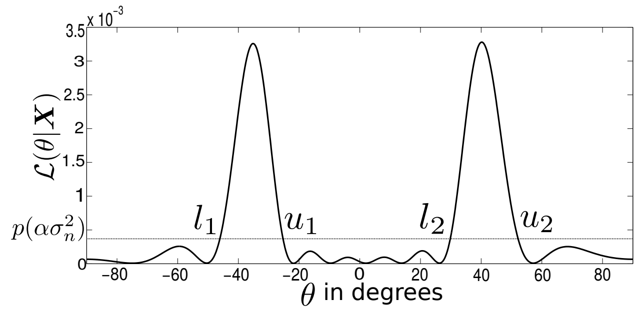

Fig. 1: $\mathcal{L}(\theta|\mathbf{X})$ for two sources

Figure 1 presents an example of the output of (14). The output of (10) is scanned for sectors, and for each sector the respective lower bound l_k and upper bound u_k are defined as shown in Figure 1. The probability floor $p(\alpha\sigma_n^2)$ defines whether a region is considered as a sector for the interpolation or not, and varies with the noise floor. The standard procedure for array interpolation would be to create a discrete set of angles within that range and employ a least squares fit. Instead of that, this work proposes the application of the UT in the likelihood within that range.

To simplify the process, the likelihood over the entire field of view must be truncated to within the detected sectors. To do that the probability left outside the sectors is distributed uniformly within the sector. This is reasonable, considering that the probability outside the sector only appears due to the presence of noise and due to harmonic terms adding up constructively in the beamformer equation. For the sake of simplicity, and since modeling spatial correlation between received signals can be extremely complex, we treat each sector as an independent likelihood. Any probability below the floor is considered to be null and the truncated total is divided within the non truncated area as

$$\int_{l_k}^{u_k} \mathcal{L}(\theta|\mathbf{X}) d\theta = 1. \tag{11}$$

The truncated continuous likelihood can be transformed into a discrete likelihood while still preserving its statistical properties by applying the UT.

Now we can apply the UT to the likelihoods defined in (11). The number of points chosen to transform each sector is a function of the number of antennas. We can define the total number of UT points $|\mathcal{S}_{\mathrm{UT}}|$ as

$$|\mathcal{S}_{\mathrm{UT}}| = M \in \mathbb{N}. \tag{12}$$

This is done so that (6) is a determined system. Thus, we have that the result from (7) is zero, that is, there are enough degrees of freedom to achieve the desired transformation over the set of angles $\mathcal{S}_{\mathrm{UT}} = [S_1, S_2, \ldots, S_M]$ defined by the application of the UT.

Figure 2 presents how the likelihood shown in Figure 1 can be discretized in the case that $M = 6$. Thus, we can construct a total transformation sector by concatenating the all the sigma points found by the UT for the detected sectors. We have that

$$\mathbf{A}_{\mathcal{S}_{\mathrm{UT}}} = [\mathbf{a}(S_1), \mathbf{a}(S_2), ..., \mathbf{a}(S_M)] \in \mathbb{C}^{M\times M}. \tag{13}$$

4.3. Transformation Matrix Calculation

A transformation that takes into account the weights found by the UT for the sigma points can be found by extending (6), resulting in

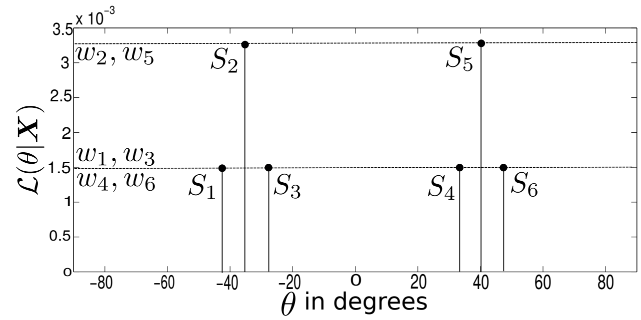

Fig. 2: UT of $\mathcal{L}(\theta|\mathbf{X})$ for two sources

$$\begin{aligned}\mathbf{B}_{UT} \quad = \quad & \bar{\mathbf{A}}_{\mathcal{S}_{\mathrm{UT}}} \sqrt{\mathrm{diag}\{[w_1, w_2, ..., w_M]\}} \mathbf{A}^{\mathrm{H}}_{\mathcal{S}_{\mathrm{UT}}} \\ & \left(\mathbf{A}_{\mathcal{S}_{\mathrm{UT}}} \sqrt{\mathrm{diag}\{[w_1, w_2, ..., w_M]\}} \mathbf{A}^{\mathrm{H}}_{\mathcal{S}_{\mathrm{UT}}}\right)^{-1} \in \mathbb{C}^{M\times M}.\end{aligned} \tag{14}$$

Following the proposed approach the transformation error shown in (7) is zero. However, the proposed approach requires the calculation of multiple sigma points through solving nonlinear systems of equations, and, for setting up such systems, it is necessary to obtain the moments of the likelihoods shown in Figure 1. In order to simplify the application and allow the proposed method to be used in real time array interpolation for systems with limited processing power, we propose calculating sigma points for common sector likelihoods, that is, defining a set of sigma points for common expected beamformer waveforms that a real time system may use.

The transformation matrix calculation can be seen as the transformation of the weighted cross covariance between the real and desired array responses into the covariance of the desired array response. However, the desired array response covariance is corrupted by the noise present in the measurements. The noise present after transformation is colored by the transformation it self, therefore, the calculation of $\mathbf{B}$ will depend on $\mathbf{B}$ itself. To solve this problem a iterative approach can be used,

$$\begin{aligned}\mathbf{B}^{(p)} \quad = \quad & \bar{\mathbf{A}}_{\mathcal{S}_{\mathrm{UT}}} \sqrt{\mathrm{diag}\{[w_1, w_2, ..., w_M]\}} \mathbf{A}^{\mathrm{H}}_{\mathcal{S}_{\mathrm{UT}}} \\ & \left(\left(\mathbf{A}_{\mathcal{S}_{\mathrm{UT}}} \sqrt{\mathrm{diag}\{[w_1, w_2, ..., w_M]\}} \mathbf{A}^{\mathrm{H}}_{\mathcal{S}_{\mathrm{UT}}}\right) + \mathbf{B}^{(p-1)} \mathbf{R}_{\mathbf{NN}} \mathbf{B}^{(p-1)\mathrm{H}}\right)^{-1} \\ & \in \mathbb{C}^{M\times M},\end{aligned} \tag{15}$$

where $\mathbf{B}^{(0)}$ can be initialized using a diagonal $\mathbf{R}_{\mathbf{NN}}$ with σ_n^2 filling its diagonal. After that a new iteration of $\mathbf{B}^{(p)}$ is obtained using the previous value until convergence.

4.4. Data Transformation

Once $\mathbf{B}$ has been obtained, the transformed covariance including FBA and SPS is obtained by [19]

$$\bar{\mathbf{R}}_{\mathbf{XX}}(L) = \frac{1}{2L} \sum_{l=1}^{L} \mathbf{J}_l^{\mathrm{T}} (\mathbf{B}\hat{\mathbf{R}}_{\mathbf{XX}}\mathbf{B}^{\mathrm{H}} + \mathbf{Q}\mathbf{B}\hat{\mathbf{R}}^{*}_{\mathbf{XX}}\mathbf{B}^{\mathrm{H}}\mathbf{Q})\mathbf{J}_l, \tag{16}$$

where $(\cdot)^*$ stands for the complex conjugation, L is the smoothing length chosen, $\mathbf{Q}$ is a matrix containing ones in its anti-diagonal and zeros elsewhere, and $\mathbf{J}_l$ is an appropriate selection matrix. After the transformation the ESPRIT algorithm can be applied as described in [29].

While L can be chosen *a priori* it can also be adaptively chosen as to minimized the loss of effective array aperture while achieving a good estimate of d. We use as a model order estimation method $\mathrm{MOE}(\bar{\mathbf{R}}_{\mathbf{XX}}(L)) = \hat{d}$. Therefore we have to solve the problem

$$(L, \hat{d}) = \arg\min_{L} \max_{\hat{d}} \left\{\mathrm{MOE}(\bar{\mathbf{R}}_{\mathbf{XX}}(L))\right\} \tag{17}$$

4.5. GEVD and ESPRIT

After solving the problem in (17) a joint high resolution estimate of the DOAs can be obtained, as shown in [37], by applying the GEVD on the FBA-SPS covariance matrix $\bar{\mathbf{R}}_{\mathbf{XX}}$.

$$\bar{\mathbf{R}}_{\mathbf{XX}}\mathbf{E} = \bar{\mathbf{R}}_{\mathbf{NN}}\mathbf{E}\boldsymbol{\Lambda}, \tag{18}$$

where $\mathbf{E} \in \mathbb{C}^{M\times M}$ is a matrix containing the generalized eigenvectors and $\boldsymbol{\Lambda} \in \mathbb{R}^{M\times M}$ is a matrix containing the generalized eigenvalues in its diagonal. By selecting the eigenvectors related to the $\hat{d}$ largest eigenvalues the so called signal subspace $\mathbf{E}_s \in \mathbb{C}^{M\times\hat{d}}$ is constructed. This signal subspace needs to be dewhitened or projected back onto the original response subspace prior to estimation, this can be done by

$$\bar{\mathbf{E}}_s = \bar{\mathbf{R}}_{\mathbf{NN}}\mathbf{E}_s. \tag{19}$$

With this subspace estimate at hand the Total Least Squares (TLS) ESPRIT [26] is applied.

5. NUMERICAL RESULT

The array response assumed in the simulations shown in Figures 4 and 5 is constructed by randomly displacing the elements of a Uniform Linear Array (ULA). The array has inner element spacing of $b = \frac{\lambda}{2}$ and the elements are displaced to a point belonging to a circle with center on the original antenna position and radius $a = \frac{0.1\lambda}{2}$, where λ is the wavelength of the carrier frequency of the signal, as shown in Figure 3 . For obtaining $\hat{\mathbf{R}}_{\mathbf{XX}}$ we use $N = 100$ snapshots and the Root Mean Squared Error (RMSE) is calculated with respect to 1000 Monte Carlo simulations. Two signals impinging from $\theta_1 = 45°$ and $\theta_2 = 15°$ with $\sigma_1^2 = \sigma_2^2 = 1$ and $\gamma_{1,2} = 1$ according to equation (4) are impinging on the array. The Signal to Noise Ratio (SNR) is defined as $\mathrm{SNR} = \frac{\sigma_1^2}{\sigma_\mathrm{n}^2} = \frac{\sigma_2^2}{\sigma_\mathrm{n}^2}$. In Figures 4 and 5 the given RMSE is

$$\mathrm{RMSE} = \sqrt{\frac{1}{K}\sum_{k=1}^{K}\left((\hat{\theta}_{1,k} - \theta_1)^2 + (\hat{\theta}_{2,k} - \theta_2)^2\right)}, \tag{20}$$

where $\hat{\theta}_{i,k}$ is the estimate of θ_i at the k-th Monte Carlo run.

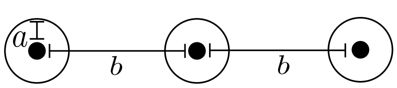

Fig. 3: Simulation array setup

Figure 4 compares the results obtained with the proposed UT approach and with the approach presented in [29] for an array length $M = 6$. SPS and FBA are applied in an adaptive manner to decorrelate the received signals. The approach adopted in [29] is capable of

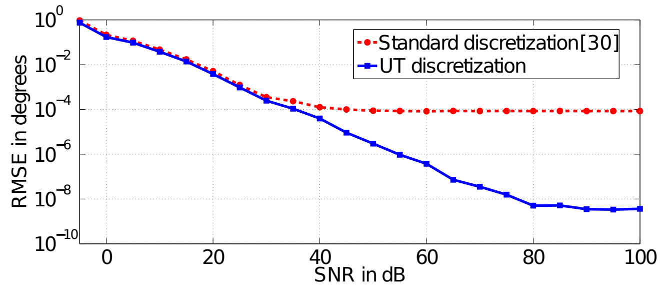

Fig. 4: Performance comparison between standard discretization and UT discretization for $M = 6$

improved accuracy as the SNR increases. However, there is a change in the inclination of the curve at $\mathrm{SNR} = 30$ dB, since the influence of the transformation error becomes greater than the influence of the noise in the final DOA estimation bias. However, since the transformation obtained with the UT is exact, there is no transformation error, and the transformation is exact for the selected points.

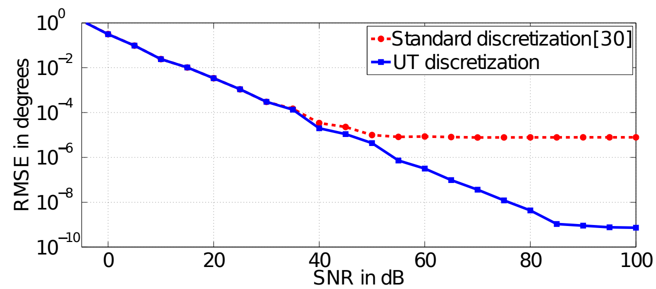

Fig. 5: Performance comparison between standard discretization and UT discretization for $M = 12$

Figure 5 compares the same approaches but for an array of length $M = 12$. The larger array results in a better RMSE for both techniques, and, since the extra number of antennas result in more degrees of freedom for the transformation, and, thus, a smaller transformation error when the standard discretization is used. Therefore, the compared techniques only begin to differ heavily in accuracy at higher SNR, where the transformation bias, albeit smaller, still results in a DOA estimation bias.

6. CONCLUSION

In this work a novel approach for array interpolation exploring the UT has been presented. In the traditional array interpolation approaches present in the literature the transformed sectors are obtained by uniformly discretazing the sector within its boundaries. This work proposed using the UT to obtain the discrete angles that better describe the signal received at the antenna array. Using the UT allows the same sectors to be described with a chosen number of angles, allowing the calculation of the transformation to be exact, reducing the bias introduced by an imperfect transformation. After transformation SPS, FBA and ESPRIT can be applied to the data.[1]

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