Reduced Rank TLS Array Interpolation for DOA Estimation

Marco A. M. Marinho*†, Felix Antreich*, João Paulo C. L. da Costa† and Josef A. Nossek‡
*German Aerospace Center (DLR), Institute for Communications and Navigation, Wessling, Germany
†Department of Electrical Engineering, University of Brasília (UnB), Brasília, Distrito Federal, Brazil
‡Inst. for Circuit Theory & Signal Processing, Munich University of Technology (TUM), Germany

***Abstract*—Important array signal processing techniques such as Spatial Smoothing, Forward Backward Averaging and Root-MUSIC require arrays with precise and specific geometries and responses. However, building sensor arrays with such demanding characteristics is not always possible. To deal with these possible limitations the real array response can be interpolated into the desired response applying array interpolation methods.**

In this work we study array interpolation methods for cases where the knowledge of the real array response is incomplete or contains errors. To address these imperfections a novel Total Least Squares (TLS) approach for calculating the transformation matrices is presented. Furthermore, a novel reduced rank regression approach is used to reduce the bias introduced by the transformation matrix onto the final direction of arrival (DOA) estimation.

***Index Terms*—array interpolation, array mapping, reduced rank regression, total least squares**

I. Introduction

Important array signal processing and direction of arrival (DOA) estimation techniques such as Spatial Smoothing (SPS) [1], Forward Backward Averaging (FBA) [2], Quadratic Maximum Likelihood (IQML) [3], Estimation of Signal Parameters via Rotational Invariance Techniques (ESPRIT) [4], Root Weighted Subspace Fitting (Root-WSF) [5], and Root-MUSIC [6] require specific array responses. IQML, SPS, Root-WSF, and Root-MUSIC demand an array response with a Vandermonde structure while the ESPRIT demands a shift invariant array response and FBA demands a centro-hermitian array response. Constructing arrays with such properties in real implementations can be very demanding and, even if possible, there is no guarantee that the response will be kept invariant over time as factors such as temperature or aging may affect the response. As a solution to these limitations the array interpolation (mapping) was first proposed in [7]. Array interpolation is used to transform the known real response of the array into a desired virtual response using a transformation matrix.

Array interpolation can be seen as a direct application of the general linear model, where the true array response represents the multivariate measurements matrix and the desired array response represents the design matrix used in this model. In the case where the real array response for a given DOA is assumed to be perfectly known and the elements of the desired array response are exogenous (fully independent from each other) the simple least squares estimation provides minimum-variance mean-unbiased estimation. Unfortunately, in practice, these assumptions usually can not be fulfilled leading to a suboptimal and biased estimation. In most cases the real array response can only be determined with empirical measurements which are often corrupted by measurement errors. While many different methods have been proposed for the calculation of a transformation matrix such as [8], [9], [10] ,[11], [12], [13], [14], [15] they all share the assumption of having perfect knowledge of the real array response.

Imperfect knowledge of the real array response will also degrade the performance of Maximum Likelihood (ML) methods [16], [17] or its extensions such as the Expectation Maximization (EM) [18], [19], [20], [21] and the space Alternating Generalized Expectation Maximization (SAGE) [22], [23], [24] since the assumed model will be mismatched with the real array response model. Furthermore, to achieve precise estimations these methods require *a-priori* knowledge of the number of signals present, when highly correlated signals are present estimating the number of incoming signals using model order selection methods such as [25], [26], [27], [28] may require the application of SPS or FBA to achieve a precise estimation. This can only be done with Vandermonde and centro-hermitian arrays, respectively.

Another common characteristic of most array interpolation methods is performing a sector-by-sector processing. Since transforming a large area of the field of view of the array can result in very large transformation errors unless the array is composed of a very large number of antennas the field of view is divided into regions called sectors. Each sector is divided into a dense and discrete set of angles used for the calculation of the transformation matrix. Since the angular separation between adjacent angles is very small the neighbor elements of the sector are highly correlated, this correlation directly impacts the estimation bias introduced by the transformation since the calculation of the transformation matrix is performed considering a highly correlated set of predictors.

In this work we present a Total Least Squares (TLS) approach for the calculation of the transformation matrix in order to deal with imperfect knowledge of the real array response. We also propose the application of a reduced rank technique for the calculation of the transformation matrix to achieve reduced bias in the final DOA estimation.

The remainder of this paper is organized in four sections. In Section II the data model used in this work is presented and detailed. Section III presents a review on the basic concepts of array interpolation and details a novel array interpolation method using TLS and rank reduction. Section IV presents a set of numerical simulations to validate the performance of the proposed method. Lastly, conclusions are drawn in Section V.

II. Data Model

We consider a set of d wavefronts impinging onto an antenna array composed of M antenna elements. The received baseband signal can be expressed in matrix form as

$$\mathbf{X} = \mathbf{A}\mathbf{S} + \mathbf{N} \in \mathbb{C}^{M \times N}, \quad (1)$$

where $\mathbf{S} \in \mathbb{C}^{d \times N}$ is the matrix containing the N symbols transmitted by each of the d sources, $\mathbf{N} \in \mathbb{C}^{M \times N}$ is the noise matrix with its entries drawn from $\mathcal{CN}(0, \sigma_n^2)$, and

$$\mathbf{A} = [\mathbf{a}(\theta_1), \mathbf{a}(\theta_2), ..., \mathbf{a}(\theta_d)] \in \mathbb{C}^{M \times d}, \quad (2)$$

where θ_i is the azimuth angle of the $i-\mathrm{th}$ signal and $\mathbf{a}(\theta_i) \in \mathbb{C}^{M \times 1}$ is the array response (empirical measurement).

The received signal covariance matrix $\mathbf{R_{XX}} \in \mathbb{C}^{M \times M}$ is given by

$$\mathbf{R_{XX}} = \mathrm{E}\{\mathbf{X}\mathbf{X}^{\mathrm{H}}\} = \mathbf{A}\mathbf{R_{SS}}\mathbf{A}^{\mathrm{H}} + \mathbf{R_{NN}}, \quad (3)$$

where $(\boldsymbol{.})^{\mathrm{H}}$ stands for the conjugate transposition, and

$$\mathbf{R_{SS}} = \begin{bmatrix} \sigma_1^2 & \gamma_{1,2}\sigma_1\sigma_2 & \cdots & \gamma_{1,d}\sigma_1\sigma_d \\ \gamma_{1,2}^*\sigma_1\sigma_2 & \sigma_2^2 & & \vdots \\ \vdots & & \ddots & \\ \gamma_{1,d}^*\sigma_1\sigma_d & \gamma_{2,d}^*\sigma_2\sigma_d & \cdots & \sigma_d^2 \end{bmatrix}, \quad (4)$$

where σ_i^2 is the power of the $i-\mathrm{th}$ signal and $\gamma_{a,b} \in \mathbb{C}$, $|\gamma_{a,b}| \leq 1$ is the cross correlation coefficient between signals a and b. $\mathbf{R_{NN}} \in \mathbb{C}^{M \times M}$ is a matrix with σ_n^2 over its diagonal and zeros elsewhere. An estimate of the signal covariance matrix can be obtained by

$$\hat{\mathbf{R}}_{\mathbf{XX}} = \frac{\mathbf{X}\mathbf{X}^{\mathrm{H}}}{N}. \quad (5)$$

III. Array Interpolation

In this section array interpolation is reviewed in Subsection III-A, a novel TLS approach is presented in III-B and a novel reduced rank model is presented in III-C.

A. Classical Array Interpolation

Array interpolation is a set of techniques that aim to predict what signal would be received at an antenna array with a specific desired geometry based on the signal that was received by a real antenna array. In matrix form the transformation tries to achieve

$$\mathbf{B}\mathbf{A}_{\mathcal{S}} = \bar{\mathbf{A}}_{\mathcal{S}}, \quad (6)$$

where $\mathbf{A}_{\mathcal{S}}$ and $\bar{\mathbf{A}}_{\mathcal{S}}$ are array response matrices constructed considering the discrete set of angles

$$\mathcal{S} = \{l_{\mathcal{S}}, l_{\mathcal{S}} + \Delta, ..., u_{\mathcal{S}} - \Delta, u_{\mathcal{S}}\}. \quad (7)$$

Here, $l_{\mathcal{S}}$ is the lower bound, $u_{\mathcal{S}}$ is the upper bound of sector $\mathcal{S}$ and Δ is the angular resolution of the transformation. The matrix $\mathbf{B}$ can be seen as the matrix that achieves the best transform between a set of vectors $\mathbf{A}_{\mathcal{S}}$ and $\bar{\mathbf{A}}_{\mathcal{S}}$. If $\mathbf{A}_{\mathcal{S}}$ is error free $\mathbf{B}$ can be obtained by a least squares (LS) fit

$$\mathbf{B} = \bar{\mathbf{A}}_{\mathcal{S}}\mathbf{A}_{\mathcal{S}}^{\dagger} \in \mathbb{C}^{M \times M}, \quad (8)$$

where $(\boldsymbol{.})^{\dagger}$ stands for the Moore–Penrose pseudo-inverse. Note that $\mathbf{B}$ is calculated differently from the usual linear regression formulation since it is obtained as to be multiplied by the right side of the original array response matrix, this is done so that $\mathbf{B}$ can then be applied directly on the estimated signal covariance matrix

$$\bar{\mathbf{R}}_{\mathbf{XX}} = \mathbf{B}\hat{\mathbf{R}}_{\mathbf{XX}}\mathbf{B}^{\mathrm{H}} \in \mathbb{C}^{M \times M}, \quad (9)$$

since

$$\begin{aligned} \mathbf{B}\mathbf{R_{XX}}\mathbf{B}^{\mathrm{H}} &= \mathbf{B}\mathbf{A}\mathbf{S}\mathbf{S}^{\mathrm{H}}\mathbf{A}^{\mathrm{H}}\mathbf{B}^{\mathrm{H}} + \mathbf{B}\mathbf{N}\mathbf{N}^{\mathrm{H}}\mathbf{B}^{\mathrm{H}} \\ &= \bar{\mathbf{A}}\mathbf{S}\mathbf{S}^{\mathrm{H}}\bar{\mathbf{A}}^{\mathrm{H}} + \mathbf{B}\mathbf{N}\mathbf{N}^{\mathrm{H}}\mathbf{B}^{\mathrm{H}}. \end{aligned} \quad (10)$$

As shown in (10) the transformation also affects the noise component, leading to colored noise at the output. This requires that some sort of prewhitening is applied prior to the DOA estimation. For prewhitening schemes, we refer to [29] for the matrix case and [30] for the tensor case.

The calculation of $\mathbf{B}$ will usually be an overdetermined one, since there will be more discrete angles in the set $\mathcal{S}$ than antennas in the virtual array. This results in an imperfect transformation, a measure of this imperfection is the transformation error given by

$$\epsilon(\mathcal{S}) = \frac{\left\|\bar{\mathbf{A}}_{\mathcal{S}} - \mathbf{B}\mathbf{A}_{\mathcal{S}}\right\|_{\mathrm{F}}}{\left\|\bar{\mathbf{A}}_{\mathcal{S}}\right\|_{\mathrm{F}}} \in \mathbb{R}^{+}. \quad (11)$$

Larger sectors will lead to larger transformation errors, and while it is possible to keep the transformation error as low as desired by keeping the sector sizes small this may lead to further problems such as demanding a very large number of estimations to be performed, one for each sector. It is possible to increase the number of antennas at the virtual array to obtain a smaller transformation error, this, however, will lead to an ill conditioned transformation matrix and to a large bias in the final DOA estimations. The number of antennas in the virtual array is usually chosen as to be equal or smaller then the number of antennas in the real array.

The problem of determining the sectors, their sizes and the resolution used for constructing the array response is a very complex optimization problem. For the purpose of this paper sectors are chosen according to [8] with $\epsilon(\mathcal{S}) < 10^{-3}$.

B. TLS Array Interpolation

In practice the real array response $\mathbf{A}_{\mathcal{S}}$ is not perfectly known. Since $\mathbf{A}_{\mathcal{S}}$ can only be estimated via empirical measurements that are corrupted by measurement errors the normal least squares approach is degraded. Although the virtual array is perfect, since it is a design variable, its output is corrupted by the noise present at the real array elements. An alternative approach is to use a regression method that takes into account errors on both sides. In this work we propose the application of the total least squares method. In this approach we calculate a intermediary transformation matrix $\tilde{\mathbf{B}} \in \mathbb{C}^{M \times M}$ using the TLS since it takes into account the presence of unknown response errors in $\mathbf{A}_{\mathcal{S}}$, such as measurement errors when obtaining the real array response empirically, represented by the matrix $\mathbf{E}$, and possible output errors, caused, for example, by the additive noise present at the antenna elements during sampling (finite sampling effect, colored noise), at $\bar{\mathbf{A}}_{\mathcal{S}}$, represented by $\mathbf{F}$. $\tilde{\mathbf{B}}$ is obtained by finding the TLS solution to

$$(\mathbf{A}_{\mathcal{S}} + \mathbf{E})^{\mathrm{H}}\tilde{\mathbf{B}} = (\bar{\mathbf{A}}_{\mathcal{S}} + \mathbf{F})^{\mathrm{H}}. \tag{12}$$

The matrices involved in (12) are arranged to allow the TLS problem to be solved using its traditional formulation. A final transformation matrix $\mathbf{B}$ can be obtained by

$$\mathbf{B} = \tilde{\mathbf{B}}^{\mathrm{H}}. \tag{13}$$

This is important since it allows the TLS solution to by applied as a right hand multiplication and achieve the same results shown in (10).

This step will only improve the results over the standard LS formulation (6) if the knowledge of $\mathbf{A}_{\mathcal{S}}$ is imperfect. On the other hand the TLS formulation is equivalent to the LS formulation when $\mathbf{A}_{\mathcal{S}}$ is perfectly known, therefore applying the TLS will never result in performance degradation when compared to the LS. The TLS approach does however come at the cost of increased computational complexity when compared to the LS formulation.

C. Reduced Rank Regression

The set $\mathcal{S}$ is composed of closely spaced angles, usually the angular resolution Δ is smaller than a degree, this will result in a high degree of correlation between the adjacent columns of the matrices $\mathbf{A}_{\mathcal{S}}$ and $\bar{\mathbf{A}}_{\mathcal{S}}$. Both the TLS and LS formulations are equivalent to regressing each column of matrices $\mathbf{A}_{\mathcal{S}}$ and $\bar{\mathbf{A}}_{\mathcal{S}}$ separately. This is a suboptimal approach since it does not take into account the relation between the responses of the closely spaced angles. One of the main problems of this approach is that it is unstable with respect to the data it is transforming. If not all the angles that compose $\mathbf{A}$ are present in $\mathcal{S}$ the results may vary drastically.

A more statistically significant and robust transformation can be obtained by performing a reduced rank regression (RR). This regression tries to transform the correlated set of predictors $\mathbf{A}_{\mathcal{S}}$ into a set of linearly independent predictors. This can be done by performing a reduced rank (RR) projection that takes into account only the principal components of $\mathbf{A}_{\mathcal{S}}$. Performing the singular-value decomposition (SVD)

$$\mathbf{B}(\mathbf{A}_{\mathcal{S}} + \mathbf{E}) = \mathbf{U}\mathbf{D}\mathbf{V}^{\mathrm{H}}, \tag{14}$$

the singular-vectors of $\mathbf{U}$ related to the r largest singular-values are selected forming $\mathbf{U}_r \in \mathbb{C}^{M\times r}$ and the reduced rank $\mathbf{B}_r$ is given by the orthogonal projection

$$\mathbf{B}_r = (\mathbf{U}_r\mathbf{U}_r^{\mathrm{H}})\mathbf{B} \in \mathbb{C}^{M\times M}. \tag{15}$$

r can be selected using a model order selection scheme, in practice we obtain a transformation matrix under a rank restriction. Note that this projection is equivalent to performing a dimension reduction of the transform matrix, since now the rows of $\mathbf{B}$ lie on a subspace in $\mathbb{C}^r$. Also note that this projection will also increase the transformation error calculate with (11) for $\mathbf{B}_r$, this, however, should not be misinterpreted as a reduction in the overall efficiency of the transformation since the final goal is not to accurately transform $\mathbf{A}_{\mathcal{S}}$ but to precisely and reliably estimate the DOAs present in $\mathbf{A}$.

Both the TLS and reduced rank projection step require the calculation of a singular-value decomposition and result in a higher extra computational burden. The extra computational complexity does not need to be included in each estimation since most array interpolation methods use a sector-by-sector processing scheme where the sectors used for the interpolation are independent of the received signal, thus the calculation of the transformation matrices can be done *a priori*.

IV. Numerical Simulations and Discussion

To study the performance of the proposed method a set of numerical simulations is performed. The performance of the proposed method is assessed in the presence of spatially white Gaussian noise and errors in the known array response. The known array response in the simulations shown in Figure 2 is constructed by randomly displacing the elements of a Uniform Linear Array (ULA) composed of $M = 8$ antennas with element spacing of $c = \frac{\lambda}{2}$ to a point on a circle with center on the original antenna position and radius $a+b = \frac{0.2\lambda}{2}$, where λ is the wavelength of the carrier frequency of the signal, to simulate errors in the known response the displacement is only known up to $a = \frac{0.1\lambda}{2}$. Figure 1 shows a graphical representation of how the antennas are displaced. While physical displacement is used in this work the same method can be used to deal with non linearity in the antenna array with respect to the DOAs of the received signals or with imperfect responses of individual antennas in the array. For each simulation run a different point is randomly chosen within the displacement circle as to avoid a displacement where all sensors are displaced in similar directions, resulting in a small relative displacement.

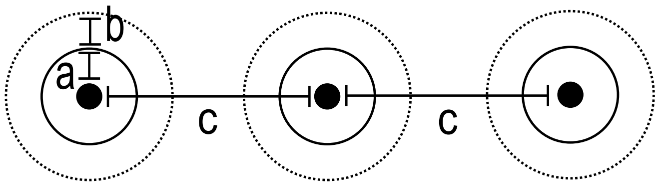

Figure 1: Graphical example of antenna displacement

$\hat{\mathbf{R}}_{\mathbf{XX}}$ is obtained using $N = 200$ snapshots and the Root Mean Squared Error (RMSE) is calculated with respect to 1000 Mont Carlo simulations. We assume three signals impinging from $\theta_1 = 45$, $\theta_2 = 38°$, and $\theta_3 = 15°$, the given RMSE is

$$\mathrm{RMSE} = \sqrt{\frac{1}{K}\sum_{k=1}^{K}\Big((\hat{\theta}_{1,k}-\theta_1)^2 + (\hat{\theta}_{2,k}-\theta_2)^2 + (\hat{\theta}_{3,k}-\theta_3)^2\Big)}, \tag{16}$$

where $\hat{\theta}_i$ is the estimate of θ_i. The Signal to Noise Ratio (SNR) is defined as

$$\mathrm{SNR} = \frac{\sigma_1^2}{\sigma_{\mathrm{n}}^2} = \frac{\sigma_2^2}{\sigma_{\mathrm{n}}^2} = \frac{\sigma_3^2}{\sigma_{\mathrm{n}}^2}.$$

To assess the performance of the proposed method under demanding conditions the set of transmitted signals is highly correlated. The wavefronts impinging from θ_1 and θ_2 are correlated with correlation coefficient $\rho = 1$ and correlated

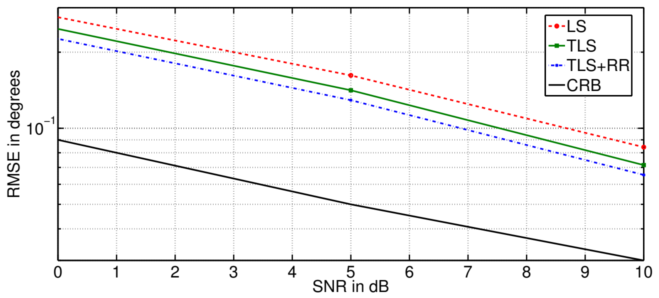

Figure 2: RMSE for classical LS, TLS and TLS+RR formulations

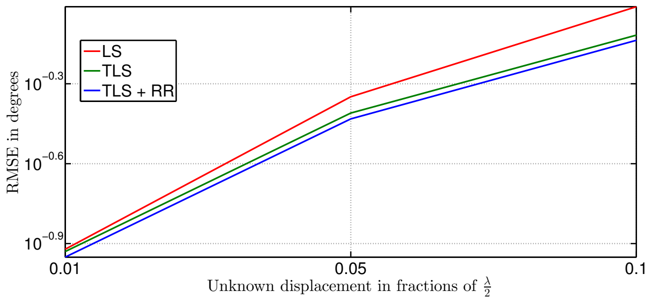

Figure 3: RMSE for classical LS and TLS versus unknown displacement

to the wavefront impinging from θ_3 with $\rho = 0.8$, the FBA-SPS approach proposed in [31] is used and DOA estimation is performed using the TLS ESPRIT and the generalized eigen-value decomposition to cope with the noise coloring introduced by the transformation.

Figure 2 presents the results when LS, only TLS, and TLS with reduced rank projection are used and the results can be compared to the Cramer Rao bound (CRB) for the real array response including errors in the array response model. The TLS formulation by itself is capable of providing an improvement of approximately 2 dB over the LS formulation. The reduced rank step is then shown to provide between 1 dB and 2 dB of extra improvement. The SPS step included in the simulations is responsible for enlarging the gap between the measured results and the CRB, since it sacrifices effective array aperture in order to decorrelate the received signals.

To compare the performance to the TLS and LS formulations with respect to the magnitude of the unknown displacement a second set of simulations was performed. For this set the SNR was kept fixed at 5 dB while the unknown displacement represent by b in Figure 1 is varied. Figure 3 shows how the unknown displacement affects the performances of the TLS and LS methods. As expected the accuracy difference between the TLS and LS increases as the unknown displacement grows.

V. Conclusion

Important DOA estimation techniques demand precise array response structures that often cannot be achieved in real implementations. To deal with these physical limitations the array interpolation method has been proposed. In practical systems the array response is often not fully known and can only be estimated within a limited degree of accuracy (empirical measurements). To deal with this problem this work proposes a novel TLS approach for the calculation of the transformation matrices that is able to cope with imperfections in the known array response. Furthermore, a novel reduced rank step is used to take advantage of the correlated nature of the array response matrices used for the calculation of the transformation matrix. The reduced rank step is shown to obtain an improved transformation that yields reduced DOA estimation bias. The computationally intensive TLS and RR steps can be performed off-line if a sector-by-sector processing approach is used, thus allowing this approach to be used in real time applications.

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